Bayesian Robustness for Fault Tree Analysis

Chaitanya Joshi
(with Fabrizio Ruggeri & S.P. Wilson)

Department of Mathematics & Statistics,
University of Waikato, New Zealand.

13th Nov 2017
BoB 2017, Gold Coast.
Motivation for this work..
Fault Tree Analysis (FTA)

- To evaluate risk in large, safety critical systems.
- To quantify the probability of occurrence of an undesirable event, called the Top event (TE).

Main assumption: Events are statistically independent.
Bayesian Networks (BN) alternative to FTA

BN can be seen as a natural extension of FTA (FTA can be directly mapped into a BN).

- Can incorporate local dependence between events.
- Enables both forward (prediction) as well as backward (inference) analysis.

However BN approaches require specifying exact prior probabilities for each elementary event and exact conditional probabilities for every dependency!

- Accurate prior probabilities are often not known.
- Computational challenges!
DePersis (2016)

- Elicit a prior distribution for each elementary event.
- Use simulations to derive the prior distributions for the intermediate events and the TE.
- Find posterior distributions using importance sampling.

Assumes that elementary events are independent.
Prior elicitation for FTA

- Eliciting the \((Beta)\) priors using expert opinion is usually not straightforward.
  - Moment matching.
  - Pairwise comparisons (using AHP)

- For very complex systems with little or no data, eliciting even a mean value (of the probability of an event) can be quite challenging for experts.

Elicited priors are likely to be inaccurate!
Prior elicitation is prone to multiple errors!

- Uncertainty due to lack of enough prior information/knowledge.
- Using insufficient or inaccurate information to elicit priors.
- Errors introduced by the methods used for prior elicitation.
- Subjectivity/bias of experts.
Prior mis-specification: snowball effect!

Consider small perturbations to the priors of each of the elementary event.

---

Chaitanya Joshi (with Fabrizio Ruggeri & S.P. Wilson) Bayesian Robustness for Fault Tree Analysis
Prior mis-specification: snowball effect!

The resulting perturbation to the prior of the TE.

This effect is especially prominent for fault trees containing OR gates.
Posterior influenced by prior

- Data on TE is often sparse and hence posterior is largely determined by prior.
  - TE is often an undesirable event - by definition unlikely to occur \((\text{hopefully})\).
  - Safety critical and/or very expensive applications, e.g. spacecraft re-entry!
  - Very little data, if any, available.

Prior mis-specification \(\approx\) posterior mis-specification.
Posterior influenced by prior

Posterior (green-dashed) vs prior (black):

Figure: (left) $n = 3$ and 1 failure and (right) $n = 10$ and 1 failure.
A distortion function

A *distortion function* $h$ is a non-decreasing continuous function $h : [0, 1] \rightarrow [0, 1]$ such that $h(0) = 0$ and $h(1) = 1$. When $h$ is used to transform the distribution function $F$,

$$F_h(X) = h \circ F(x) = h[F(x)]$$

represents a perturbation of $F$ in order to measure the uncertainty about it. Note that $F_h(X)$ is also a distribution function for a particular random variable denoted by $X_h$ and the distorted density is given by

$$f_h(X) = h'[F(x)] \cdot f(x).$$
Stochastic ordering

For two random variables $X$ and $Y$, $X$ is said to be smaller than $Y$ in the stochastic order sense (denoted by $X \leq_{st} Y$) if

$$F_X(t) \geq F_Y(t), \forall t \in \mathbb{R}.$$ 

For absolutely continuous [discrete] random variables $X$ and $Y$ with densities [discrete densities] $f_X$ and $f_Y$, respectively, $X$ is said to be smaller than $Y$ in the likelihood ratio order sense (denoted by $X \leq_{lr} Y$) if

$$\frac{f_Y}{f_X} \text{ increases over the union of the supports of } X \text{ and } Y.$$ 

It is well known that

$$X \leq_{lr} Y \Rightarrow X \leq_{st} Y.$$
Convex and concave distortion functions

- If $\pi$ is a specific prior belief with distribution function $F_{\pi}$ and $h$ is a convex (concave) distortion function in $[0, 1]$, then $\pi \leq_{lr} (\geq_{lr}) \pi_h$.

- If the decision maker is able to represent the changes to a prior belief $\pi$ by a concave distortion function $h_1$ and a convex distortion function $h_2$, then it leads him to two distorted distributions $\pi_{h_1}$ and $\pi_{h_2}$ such that $\pi_{h_1} \leq_{lr} \pi \leq_{lr} \pi_{h_2}$.

- This defines the class of priors called the **distorted band of priors** $\Gamma_{h_1,h_2,\pi}$ as

$$\Gamma_{h_1,h_2,\pi} = \{ \pi' : \pi_{h_1} \leq_{lr} \pi' \leq_{lr} \pi_{h_2} \}.$$  \hspace{1cm} (1)

Arias-Nicolás et al. (2016).
Distortion bands for a prior distribution

Elicited prior (*black*), lower (*green*) and upper (*red*) distortion bands.
Power functions as distortion functions

A popular choice for distortion functions $h_1$ and $h_2$ are power functions given by

$$h_1(x) = 1 - (1 - x)^\alpha \text{ and } h_2(x) = x^\alpha, \forall \alpha > 1.$$  \hspace{1cm} (2)

Note that if we take $\alpha = n \in \mathbb{N}$ in (2), then $F_{\pi h_1}(\theta) = 1 - (1 - F_\pi(\theta))^n$ and $F_{\pi h_2} = (F_\pi(\theta))^n$ which correspond to the distribution functions of the minimum and the maximum, respectively, of an i.i.d. random sample of size $n$ from the baseline prior distribution $\pi$.

- Power functions are easily used in applications and also give interesting results.
The Kolmogorov metric measures the maximum absolute difference between the two distribution functions and is defined by

$$K(X, Y) = \sup_{x \in \mathbb{R}} |F_X(x) - F_Y(x)|.$$  

Kolmogorov metrics between elicited prior and its distortions are $K(\pi, \pi_{h_1})$ and $K(\pi, \pi_{h_2})$

Interpretation: *How far off could the true prior be from the elicited one in the worst case?*  
No more than 20% $\Rightarrow K = 0.2$
If the distortion functions are defined as in (2) then, the Kolmogorov metric is given by the following expression (Arias-Nicolás et al. (2016)):

\[ K(\pi, \pi_{h_1}) = K(\pi, \pi_{h_2}) = \frac{\alpha - 1}{\alpha^{-1} \sqrt{\alpha}}. \]  

(3)

Equation (3) can be used for eliciting \( \alpha \).

- Given \( K \), find \( \alpha \) using a computer program using (3).
- Alternatively, a rough estimate of \( \alpha \) can be obtained assuming \( \alpha^{-1} \sqrt{\alpha} \approx \alpha \) as

\[ \alpha \approx \frac{1}{1 - K}. \]
Power functions as distortion functions

For distorted bands obtained using power functions, one can show that:

- $\alpha$ controls the width of the distortion band in a strict monotonic way (so a distorted band obtained using a smaller $\alpha$ is completely contained inside the band obtained using a larger $\alpha$).
- Stochastic order and likelihood order are equivalent.

**Theorem**

When the distortion functions are defined as in (2),

(i) $X \leq_{st} Y \iff X \leq_{lr} Y$, (ii) $1 \leq \alpha_1 \leq \alpha_2 \Rightarrow \Gamma_{\alpha_1} \subset \Gamma_{\alpha_2}$ and (iii) $\Gamma_{\alpha} \rightarrow F(x)$ as $\alpha \downarrow 1$. 

Chaitanya Joshi (with Fabrizio Ruggeri & S.P. Wilson)  Bayesian Robustness for Fault Tree Analysis
Bayesian robustness for FTA

Given that a prior distribution has been elicited for each of the elementary events.

Bayesian robustness for FTA - an outline

Step I: Build a distorted band of priors for each event.

Step II: Simulate through the FT using algorithms A1 - A4 to find the prior distribution and the distorted band of priors for the intermediate events and the TE.

Step III: Find the posterior distribution for the TE given the prior distribution and the data.

Step IV: Find the lower and the upper distortion bands for the posterior distribution of the TE given the distorted bands for the prior and the data.
Bayesian robustness for FTA

**Step I:** Build a distorted band of priors for each event $i$.

- Assume that the prior distribution $\pi_i$ has been elicited for each of the elementary events.
- Elicit $K$ and therefore $\alpha$ using Equation (3).
- Determine the lower bound $\pi_{h_1i}$ using the concave $h_{1i}$ in 2.
- Determine the upper bound $\pi_{h_2i}$ using the convex $h_{2i}$ in 2.
Bayesian robustness for FTA

Step II: Simulate through the FT using algorithms A1 - A4 to find the prior distribution and the distorted band of priors for the intermediate events and the TE.

- **Algorithm A1**: to simulate prior distributions for intermediate and top events.
- **Algorithm A2**: to simulate distortion bands for the prior distributions for intermediate and top events.
- **Algorithm A3**: to simulate from $\pi_{h_1i}$ for $h_1i$ concave.
- **Algorithm A4**: to simulate from $\pi_{h_2i}$ for $h_2i$ convex.
Step II: Algorithms A3 and A4 are rejection sampling based algorithms making use of the fact that $h_{1i}$ ($h_{2i}$) is concave (convex) and hence has a derivative that is monotonically decreasing (non-decreasing).

Algorithm A3: to simulate from $\pi_{h_{1i}}$ for $h_{1i}$ concave

1. Sample $\theta_{ij} \sim \pi_i(\theta_i)$, $j = 1, \ldots, N$, $i = 1, 2, 3$ and $u_j \sim U(0, 1)$ independently.

2. For each $j$, check if $u_j \leq \frac{h'_{1i}[F_i(\theta_{ij})]}{h'_{1i}[0]}$
   - If this holds, accept $\theta_{j}$ as a realisation of $\pi_{h_{1i}}$.
   - If not, reject the value $\theta_{ij}$. 
Step II: Algorithms A2 assumes that it is sufficient to sample from \( \pi_{h_1} \)'s to obtain \( \pi_{h_1} \) and to sample from \( \pi_{h_2} \)'s to obtain \( \pi_{h_2} \). It can be proven that this assumption is indeed valid.

**Theorem**

In order to obtain the distorted lower (upper) bands for the intermediate/top event by sampling from them, it is necessary and sufficient to sample only from the respective lower (upper) bands of the elementary events.
Steps III and IV: Posterior distribution and the distorted band for the posterior distribution are obtained using the importance sampling algorithm by DePersis (2016).

- Proposal distribution - prior distribution of TE.
- Importance weights using the likelihood.
Example: Spacecraft re-entry

<table>
<thead>
<tr>
<th>Event</th>
<th>Description</th>
<th>Event</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE$</td>
<td>Explosion of the spacecraft</td>
<td>$E_{13}$</td>
<td>Chemical reactions</td>
</tr>
<tr>
<td>$E_{21}$</td>
<td>Chemical reaction of propellant and air</td>
<td>$E_{14}$</td>
<td>Over pressure</td>
</tr>
<tr>
<td>$E_{22}$</td>
<td>Burst of pressure vessel</td>
<td>$E_{15}$</td>
<td>Short circuit</td>
</tr>
<tr>
<td>$E_{23}$</td>
<td>Chemical reaction between hypergolic propellants</td>
<td>$E_{16}$</td>
<td>Corrosion</td>
</tr>
<tr>
<td>$E_{24}$</td>
<td>Burst of battery cell</td>
<td>$E_{17}$</td>
<td>Over charge</td>
</tr>
<tr>
<td>$E_{11}$</td>
<td>Sudden release of propellant ($E_{22}$)</td>
<td>$E_{18}$</td>
<td>Over discharge</td>
</tr>
<tr>
<td>$E_{12}$</td>
<td>Slow release of propellant</td>
<td>$E_{19}$</td>
<td>Over temperature</td>
</tr>
<tr>
<td>$E_{01}$</td>
<td>Valve leakage</td>
<td>$E_{110}$</td>
<td>Cell degradation</td>
</tr>
<tr>
<td>$E_{02}$</td>
<td>Tank destruction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_{03}$</td>
<td>Pipe rupture</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Chaitanya Joshi (with Fabrizio Ruggeri & S.P. Wilson)
Example: Spacecraft re-entry

Figure: [a] The fault tree used to model the spacecraft re-entry problem and [b] the simplified fault tree in minimum cut-set
Example: Spacecraft re-entry

<table>
<thead>
<tr>
<th>Event</th>
<th>Weight</th>
<th>Range</th>
<th>Elicited prior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{22}$</td>
<td>0.83333</td>
<td>(0.01, 0.05)</td>
<td>Beta(6.3,233) *</td>
</tr>
<tr>
<td>$E_{23}$</td>
<td>0.16667</td>
<td>(0.002, 0.01)</td>
<td>Beta(6.4,1214)</td>
</tr>
<tr>
<td>$E_{01}$</td>
<td>0.42857</td>
<td>(0.01, 0.04)</td>
<td>Beta(8.3,360)*</td>
</tr>
<tr>
<td>$E_{02}$</td>
<td>0.1428</td>
<td>(0.0033, 0.0133)</td>
<td>Beta(8.3,1104)</td>
</tr>
<tr>
<td>$E_{03}$</td>
<td>0.42857</td>
<td>(0.01, 0.04)</td>
<td>Beta(8.3,360)</td>
</tr>
<tr>
<td>$E_{13}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)*</td>
</tr>
<tr>
<td>$E_{14}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
<tr>
<td>$E_{15}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
<tr>
<td>$E_{16}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
<tr>
<td>$E_{17}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
<tr>
<td>$E_{18}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
<tr>
<td>$E_{19}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
<tr>
<td>$E_{110}$</td>
<td>0.125</td>
<td>(0.014, 0.055)</td>
<td>Beta(8.4,261)</td>
</tr>
</tbody>
</table>

Table: Elicited priors obtained using the AHP process. * indicates that the prior was elicited using the range provided by the expert.
Example: Spacecraft re-entry

Likelihood:

- $TE$ corresponds to whether the spacecraft exploded ($TE = 1$) or not ($TE = 0$) during the re-entry.

- $TE \sim Bernoulli(\theta_{TE})$, where $\theta_{TE} = 1 - \prod_j (1 - \theta_j)$.

- If the data was obtained from $n$ identical spacecraft re-entries then $TE \sim Binomial(n, \theta_{TE})$.

- We assume that only the top event is observed and that none of the elementary events are directly observed.

Distortion bands: We assume that $K = 0.15$ and obtain $\alpha = 1.51$ using Equation 3.
Example: Spacecraft re-entry

Figure: (Top left) the unique prior distributions in Table 1. (Remaining) each of the priors and the distorted bands obtained - lower band in green - dotted and upper band in red - dashed.
Example: Spacecraft re-entry

Figure: (Left) The prior distribution of $\theta_{TE}$ and its distortion bands, (right) the posterior distribution of $\theta_{TE}$ and its distortion bands: lower band in green - dotted and upper band in red - dashed.

Chaitanya Joshi (with Fabrizio Ruggeri & S.P. Wilson) Bayesian Robustness for Fault Tree Analysis
Summary

• This work:
  • Shows how distortion bands obtained using power functions can be used in Bayesian FTA approaches.
  • Provides the sampling algorithms to implement Bayesian FTA.

• Prior robustness study essential!
  • Distortion bands (Arias-Nicolás et al. 2016) have many practical advantages.

• Further/ current work:
  • Prior robustness for ABC methods.